arXiv:hep-th/9612167v1 16 Dec 1996

On higher-dimensional loop algebras, pseudodifferential operators and Fock space realizations

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We discuss a previously discovered [1] extension of the infinite-dimensional Lie algebras $\mathrm{Map}(M, \mathfrak{g})$ which generalizes the Kac–Moody algebras in 1+1 dimensions and the Mickelsson–Faddeev algebras in 3+1 dimensions to manifolds M of general dimensions. Furthermore, we review the method of regularizing current algebras in higher dimensions using pseudodifferential operator (PSDO) symbol calculus. In particular, we discuss the issue of Lie algebra cohomology of PSDOs and its relation to the Schwinger terms arising in the quantization process. Finally, we apply this regularization method to the algebra of ref. [1] with partial success, and discuss the remaining obstacles to the construction of a Fock space representation.

1. Introduction

The importance of infinite-dimensional Lie algebras and their representation theory for two-dimensional quantum field theory has become overwhelmingly clear over the past two decades. A prime example of such algebras are, of course, the affine Kac–Moody (KM) algebras, which are realized in physical models as extensions of loop algebras $\mathrm{Map}(S^1, \mathfrak{g})$ or as two-dimensional current algebras, and as such allow for generalizations to higher dimensional systems.

However, the usefulness of these higher-dimensional analogues is hampered by their poorly developed representation theory; generalizing the techniques and methods used in two dimensions has turned out to be a very difficult task. The most extensively studied case in this respect is probably the Mickelsson–Faddeev (MF) algebras [2,3], which arise as anomalous constraint algebras in chiral gauge theory.

Here we shall describe an attempt at obtaining interesting representations of a particular extension of the algebras $\mathrm{Map}(M_d, \mathfrak{g})$ of maps from a d-dimensional manifold M_d to a finite-dimensional Lie algebra $\mathfrak{g}$ that was discovered [1] in the context of p-branes. The primary reason for the difficulties encountered in the study of higher-dimensional current algebras is

the additional regularization needed to obtain one-particle Hilbert space operators that by a standard normal-ordering procedure can be elevated to Fock space operators furnishing a unitary highest weight (HW) representation of the algebra. Following Mickelsson [4], we will employ PSDO symbol calculus to perform this regularization. After a presentation of the algebra in sect. 2 we review some necessary backgrund material on fermionic Fock space constructions of current algebras and on PSDOs in Sects. 3 and 4, respectively. In sect. 5 we present our main results, and finally end the paper by some comments in sect. 6.

2. Loop algebras in general dimensions

In ref. [1] we presented the following extension of $\mathrm{Map}(M_d, \mathfrak{g})$:

$$[\![(X; z), (Y; w)]\!]$$
$$= ([X, Y] + [dX, dY]; \alpha_d \int_{M_d} \mathrm{tr}\, X\, dY) \qquad (1)$$

Here $X = \bigoplus_{i=0}^{[d/2]} X^{(2i)}$ and $Y = \bigoplus_{i=0}^{[d/2]} Y^{(2i)}$ are sums of differential forms of even degree taking values in the compact Lie algebra $\mathfrak{g}$, while $z, w \in \mathbb{C}$ and α_d is a normalization constant. The brackets on the right hand side are ordinary (not graded) commutators. Furthermore, wedge products between forms are understood and forms of

degree higher than $d = \dim M_d$ are assumed to vanish. The integral, which defines the central term of the algebra, selects the terms in the form expansion proportional to the volume form.

The characteristic features of these algebras are that their formulation is diffeomorphism invariant, valid for general d and reduces to the well-known KM and MF algebras for $d = 1, 3$, respectively. Furthermore, in contrast to the MF algebras in general dimensions [3], the algebras (1) are always linear. We should also mention that (1) defines a non-abelian extension of $\mathrm{Map}(M_d, \mathfrak{g})$ when $d \geq 4$ [1].

The algebra (1) may alternatively be formulated as an algebra of smeared current densities:

$$[T(X), T(Y)] = T([X, Y] + [dX, dY]) + k\,\alpha_d \int_{M_d} \mathrm{tr}\, X\, dY \tag{2}$$

In this form, the algebra was used in ref. [1] as the starting-point of a p-loop space formulation of a model for odd-dimensional p-branes coupled to a background Yang–Mills field; by imposing (2) as the algebra obeyed by certain functional operators appearing in the BRST transformations, it was found to be possible to construct a nilpotent BRST operator, as well as a p-loop space gauge covariant derivative and the corresponding curvature tensor. This construction was an attempt to get round the difficulties caused by non-linearities arising in a corresponding formulation [5,6] based on a Kaluza–Klein type p-brane action [7]. Although, so far, we have not been able to derive our algebra from any particular p-brane action, recent developments in the subject of Dirichlet p-branes (D-branes) gives reason for some optimism in this respect. For instance, similar geometrical constructions involving sums of differential forms of various degrees have been found very useful in the construction of effective world-volume actions for D-branes [8–12].

3. Abstract current algebras

We now wish to look for unitary HW representations of the above algebra on a fermionic Fock space. To this end, let us first review the general framework for the study of current algebras that

we will make use of [13,14].

Thus, consider a one-particle description of free Dirac or Weyl fermions $\psi \in \mathcal{H} = \mathrm{L}^2(\mathbb{R}^d) \otimes V = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ ($\mathcal{H}_-$) denotes the positive (negative) energy subspace as determined by the hamiltonian H, and V is a finite-dimensional vector space carrying the internal (spin and "color") degrees of freedom. This quantum mechanical model is second-quantized by introducing a Fock vacuum $|0\rangle \in \mathcal{F}(\mathcal{H})$ such that

$$a_n^*|0\rangle = 0,\ n < 0, \qquad a_n|0\rangle = 0,\ n \geq 0, \tag{3}$$

where the creation and annihilation operators obey the canonical anticommutation relations

$$\{a_m^*, a_n\} = \delta_{mn},\ \{a_m^*, a_n^*\} = \{a_m, a_n\} = 0. \tag{4}$$

The second-quantized observables $\hat{A}$ acting on $\mathcal{F}(\mathcal{H})$ are defined by the linear map

$$A \longmapsto \hat{A} = \sum_{m,n} A_{mn} : a_m^* a_n :, \tag{5}$$

with the normal-ordering

$$: a_m^* a_n := \begin{cases} -a_n a_m^* & \text{if } m, n < 0 \\ a_m^* a_n & \text{otherwise} \end{cases} \tag{6}$$

defined to give $\hat{A}$ a vanishing vacuum expectation value.

In order for $\hat{A}$ to be a well-defined operator on $\mathcal{F}(\mathcal{H})$ it has to obey the crucial condition

$$\|\hat{A}|0\rangle\| < \infty, \tag{7}$$

which may readily be reformulated as $A \in \mathfrak{gl}_1(\mathcal{H})$, where for $q \in \mathbb{Z}_+$ the general linear algebras of $\mathcal{H}$ are defined as

$$\mathfrak{gl}_q(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \,|\, [\varepsilon, A] \in \mathcal{L}_{2q}(\mathcal{H})\}. \tag{8}$$

Here $\varepsilon = \mathrm{sign}\, H$, $\mathcal{B}(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$, and the Schatten ideals $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_2(\mathcal{H}) \subset \dots$ are defined by

$$\mathcal{L}_p(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \,|\, \mathrm{Tr}\, |A^* A|^{p/2} < \infty\}, \tag{9}$$

with Tr denoting the Hilbert space trace. In particular, operators in $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{L}_2(\mathcal{H})$ are called trace-class and Hilbert–Schmidt (HS), respectively.

As it turns out, the Fock space operators obtained by second-quantizing $\mathfrak{gl}_1(\mathcal{H})$ do not satisfy the same Lie algebra as their one-particle ancestors. Instead one finds the commutation relations

$$[\hat{A}, \hat{B}] = \widehat{[A, B]} + c_1(A, B), \qquad (10)$$

defining the universal central extension $\widehat{\mathfrak{gl}_1(\mathcal{H})}$ of $\mathfrak{gl}_1(\mathcal{H})$. The Schwinger term

$$c_1(A, B) = \frac{1}{4}\text{Tr}\,\varepsilon[\varepsilon, A][\varepsilon, B] \qquad (11)$$

of this current algebra is a non-trivial Lie algebra two-cocycle originally found by Lundberg [13][1].

Since $\widehat{\mathfrak{gl}_1(\mathcal{H})}$ has unitary HW representations [15], a basic strategy for constructing such representations of various extensions of $\text{Map}(M_d, \mathfrak{g})$ is to embed the latter in $\widehat{\mathfrak{gl}_1(\mathcal{H})}$. On the one-particle level there is a natural embedding $\text{Map}(M_d, \mathfrak{g}) \hookrightarrow \mathcal{B}(\mathcal{H})$ as multiplicative operators:

$$X \mapsto \tilde{X}: \ \tilde{X}\psi(x) = X(x)\psi(x), \ x \in M_d. \qquad (12)$$

However, due to ultra-violet divergencies such operators belong to $\mathfrak{gl}_1(\mathcal{H})$ iff $d = 1$; in general $\text{Map}(M_d, \mathfrak{g}) \hookrightarrow \mathfrak{gl}_q(\mathcal{H})$ only for $q > d/2$. For $M_d = S^1$ one obtains in this way spinor representations of affine KM algebras [15]. When $d > 1$ on the other hand, normal-ordering is not sufficient to yield well-defined second-quantized operators. One method that has been employed to deal with this problem is to enlarge the Fock space by considering current algebras based on $\mathfrak{gl}_q(\mathcal{H})$ for $q > 1$ [16–18]. However, it has been shown [19] that $\widehat{\mathfrak{gl}_2(\mathcal{H})}$, which is the relevant algebra for $d = 3$, has no unitary HW representations.

An alternative strategy is to perform additional regularization of the one-particle observables before second-quantizing. For the MF algebra in $d = 3$ this approach leads to projective representations in terms of one-cocycles valued in the space of functionals of the gauge fields [20]. A convenient way to perform the regularization [4] is by means of PSDO symbol calculus, since

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the UV behavior can then be characterized and treated very explicitly. In sect. 5 we will apply this method to the algebra discussed in sect. 2, but first we will give the prerequisites from the theory of PSDOs.

4. PSDOs and their cohomology

Consider the Hilbert space $\mathcal{H} = \text{L}^2(\Omega_d) \otimes \mathbb{C}^N$ of square-integrable vector-valued functions with support on some compact domain $\Omega_d \subset \mathbb{R}^d$ A pseudodifferential operator S on $\mathcal{H}$ is defined via its symbol $s : \Omega_d \times \mathbb{R}^d \to \mathfrak{gl}(N, \mathbb{C})$ by

$$S\psi(x) = \frac{1}{(2\pi)^d} \int s(x, p)\tilde{\psi}(p)e^{ix \cdot p}d^dp, \qquad (13)$$

where $\tilde{\psi}(p) = \int \psi(x)e^{-ix \cdot p}d^dx$ is the Fourier transform of $\psi \in \mathcal{H}$. Here we shall consider only smooth symbols of integral order, the order $\text{Ord}\,(s) = m$ of a symbol s being defined in terms of its leading asymptotic behavior $s(x, p) = \mathcal{O}(|p|^m)$ for large $|p|$. We denote the space of such symbols as Sym^m and the corresponding space of PSDOs as Ψ^m. In particular, S is called an infinitely smoothing operator ($S \in \Psi^{-\infty}$) if its symbol vanishes more rapidly than any power of p as $|p| \to \infty$. Furthermore, two PSDOs S and S' are called equivalent, denoted as $S \approx S'$ (or $s \approx s'$ for their symbols), if $S - S' \in \Psi^{-\infty}$. Up to equivalence, a PSDO $S \in \Psi^m$ with symbol s is determined by an asymptotic expansion

$$s(x, p) \approx \sum_{k \leq m} s_k(x, p), \qquad (14)$$

where each term s_k is taken to be smooth and homogeneous of degree k in p outside some finite radius.

Composition of PSDOs is defined on the equivalence classes $\text{Sym}^\infty/\text{Sym}^{-\infty}$ by the star product

$$\sigma(S) * \sigma(S') \approx \sigma(SS'), \qquad (15)$$

where $S \in \Psi^m$, $S' \in \Psi^n$ and $\sigma : \Psi^m \to \text{Sym}^m$ denotes the symbol map. The asymptotic behavior of the star product $s * s' \in \text{Sym}^{m+n}$ of $s \in \text{Sym}^m$ and $s' \in \text{Sym}^n$ is

$$(s * s')(x, p) \approx \sum_{k=0}^\infty \frac{(-i)^k}{k!} \frac{\partial^k s}{\partial p_{i_1}...\partial p_{i_k}} \frac{\partial^k s'}{\partial x^{i_1}...\partial x^{i_k}}.\,(16)$$

A PSDO S is bounded iff $S \in \Psi^0$ and belongs to the Schatten ideal $\mathcal{L}_p(\mathcal{H})$ iff $S \in \Psi^m$ with $m < -d/p$. The latter result is easily established by taking into account the asymptotic behavior $\mathcal{O}(|q|^m)$ and using the expression[2]

$$\operatorname{Tr} S = \int_{\Omega_d \times \mathbb{R}^d} \operatorname{tr} s(x,p)\, d^d x\, d^d p \qquad (17)$$

for the Hilbert space trace of a PSDO $S \in \Psi^k$ with symbol s. Hence, S is trace-class iff $\operatorname{Ord} S < -d$ and Hilbert–Schmidt iff $\operatorname{Ord} S < -d/2$. A property of Tr that will be of central importance below, is that it is not well defined on the equivalence classes of trace-class PSDOs. This follows immediately from the observation that Tr does not vanish on $\Psi^{-\infty}$. Moreover, we can notice that by calculating the finite-dimensional trace and the integral in the proper order the class of PSDOs which yield a finite result can be extended; more specifically, one can define a conditional trace [21] by

$$\operatorname{Tr_c} S = \lim_{\Lambda \to \infty} \int_{|p| \le \Lambda} \frac{d^d p}{(2\pi)^d} \int_{\Omega_d} d^d x\, \operatorname{tr} s(x,p), \quad (18)$$

with the integrations performed in the order indicated. Operators for which (18) converges are called conditionally trace-class. Of course, for trace-class PSDOs Tr and $\operatorname{Tr_c}$ coincide.

In contrast to Tr and $\operatorname{Tr_c}$, the Wodzicki residue [22,23]

$$\operatorname{Res}(s) = \frac{1}{(2\pi)^d} \int_{|p|=1} d\Omega_p \int_{\Omega_d} d^d x\, \operatorname{tr} s_{-d}(x,p) \quad (19)$$

gives a well-defined (and unique) trace-functional on $\Psi^\infty / \Psi^{-\infty}$. Here s_{-d} is the coefficient of order $-d$ in an asymptotic expansion for s. The Wodzicki residue can in turn be used to define a non-trivial two-cocycle on the Lie algebra of PSDOs known as the Radul cocycle [24]:

$$c_{\mathrm{R}}(S,S') = -\frac{1}{2}\operatorname{Res}([\log|p|,s]_* * s'). \qquad (20)$$

Here $[.,.]_*$ denotes the star commutator defined in the obvious way.

So far the choice of Hilbert space $\mathcal{H}$ has been fairly arbitrary. To make contact with the framework of sect. 3 we will from this point on consider

Weyl fermions in $d = 2n - 1$ dimensions with $\mathcal{H} = \mathrm{L}^2(\Omega_d) \otimes \mathbb{C}^{n-1}_{\mathrm{spin}} \otimes \mathbb{C}^N_{\mathrm{color}}$. Equipped with the grading operator $\varepsilon = \operatorname{sign} H$, where $H = -i\partial\!\!\!/$ is the free Weyl Hamiltonian, we can embed the PSDOs in the general linear algebras $\mathfrak{gl}_q$. Recalling the definition (8) and the Schatten ideal conditions above we find, in particular, that a PSDO S belongs to $\mathfrak{gl}_1$, which we write as $S \in \Psi\mathfrak{gl}_1$, iff $\operatorname{Ord}[\varepsilon,S] < -d/2$ [3]. In terms of the corresponding symbols this reads

$$S \in \Psi\mathfrak{gl}_1 \Leftrightarrow \operatorname{Ord}[\varepsilon,s]_* < -\frac{d}{2}, \qquad (21)$$

where we have used the notation ε also for the symbol of the grading operator. When expanding the star commutator, this condition translates into equations relating the coefficients of an asymptotic expansion for s which can be solved order by order, thus yielding PSDOs with a good second-quantized behavior.

Turning to the second-quantization, one immediately has to face the problem that the Lundberg cocycle c_1 entering in the algebra $\widehat{\mathfrak{gl}_1}$ of the second-quantized operators is not well defined on the equivalence classes $\Psi\mathfrak{gl}_1/\Psi^{-\infty}$ of second-quantizable PSDOs due to the fact that it is defined in terms of the Hilbert space trace Tr [4]. Nevertheless, we have shown in [25] that the restriction of the Lundberg cocycle to $\Psi\mathfrak{gl}_1$, defined in terms of the conditional trace (18), is cohomologically equivalent to a "twisted" version [26]

$$\tilde{c}_{\mathrm{R}}(S,S') = -\frac{1}{2}\operatorname{Res}(\varepsilon * [\log|p|,s]_* * s') \qquad (23)$$

of the Radul cocycle. In other words, for any two operators $S, S' \in \Psi\mathfrak{gl}_1/\Psi^{-\infty}$, defined by their asymptotic expansions,

$$c_1(S,S') - \tilde{c}_{\mathrm{R}}(S,S') = \delta\lambda(S,S'), \qquad (24)$$

where δ is the Lie algebra coboundary operator and λ is a certain one-cochain on $\Psi\mathfrak{gl}_1/\Psi^{-\infty}$. For

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[2] Here tr denotes the finite-dimensional matrix trace.

[3] We will refer to this as the Hilbert–Schmidt condition.

[4] Actually, it turns out that the proper trace to use when defining the Schatten ideals, and hence also the Lundberg cocycle, is not Tr but the conditional trace which in general may be defined as

$$\operatorname{Tr_c}(A) = \frac{1}{2}\operatorname{Tr}(A + \varepsilon A \varepsilon). \qquad (22)$$

further details and discussion the reader is referred to ref. [25].

Notice that this is a rather general result independent of any particular algebra which one might try to embed in $\Psi\mathfrak{gl}_1/\Psi^{-\infty}$. Next we shall use it in the second-quantization of the algebra (1).

5. Fock space realization

Finally, we are ready to consider the issue of Fock space realizations of the algebras (1). We shall consider the case $M_d = \mathbf{R}^3$, as it turns out that this is the highest dimension in which the regularization method works (at least partially) for the particular algebras (1). Preliminary results on this construction were reported in ref. [27].

Define the maps $\rho : \mathcal{L}_3 \to \Psi\mathfrak{gl}_1$ and $\bar{\rho} = \sigma \circ \rho$, where σ is the symbol map, by the asymptotic expansions

$$\bar{\rho}(X^{(0)}) = X^{(0)} + S^i \frac{\partial X^{(0)}}{\partial x^i} + \frac{1}{2} S^{ij} \frac{\partial^2 X^{(0)}}{\partial x^i \partial x^j} + \cdots \quad (25)$$

$$\bar{\rho}(X^{(2)}) = \frac{1}{2} A^{ij} X^{(2)}_{ij} + \frac{1}{6} A^{ijk} \partial_i X^{(2)}_{jk} + \cdots \quad (26)$$

Here S^i is of order -1, S^{ij} and A^{ij} of order -2, etc. Moreover, we have introduced the notation $\mathcal{L}_3$ for the Lie algebra

$$[\![X, Y]\!] = [X, Y] + [dX, dY]. \quad (27)$$

Its central extension $\hat{\mathcal{L}}_3$ defined by the two-cocycle

$$\omega_3(X, Y) = -\frac{1}{6\pi} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \operatorname{tr} X \, dY \quad (28)$$

is then the algebra (1) under consideration.

We impose two conditions on the map $\bar{\rho}$:

$$\operatorname{Ord}[\varepsilon, \bar{\rho}(X)]_* \leq -2, \quad (29)$$

$$[\bar{\rho}(X), \bar{\rho}(Y)]_* = \bar{\rho}([X, Y] + [dX, dY]). \quad (30)$$

The first one is the Hilbert–Schmidt condition ensuring that $\rho(X) \in \Psi\mathfrak{gl}_1$; the second is the homomorphism condition. Inserting (25) in (29) determines S^i. Using also (26) and (30), S^{ij} and A^{ij} can then be expressed in terms of S^i. Explicitly,

we find ($\varepsilon = p\!\!\!/\,/|p| = p^i \sigma_i/|p|$)

$$S^i = \frac{i}{2} \varepsilon \frac{\partial \varepsilon}{\partial p_i} = \frac{1}{2p^2} \epsilon^{ijk} p_j \sigma_k,$$

$$S^{ij} = \frac{1}{2}(S^i S^j + S^j S^i) - \frac{i}{2}\left(\frac{\partial S^i}{\partial p_j} + \frac{\partial S^j}{\partial p_i}\right), \quad (31)$$

$$A^{ij} = \frac{1}{2}(S^i S^j - S^j S^i) - \frac{i}{2}\left(\frac{\partial S^i}{\partial p_j} - \frac{\partial S^j}{\partial p_i}\right).$$

At order ≤ -3, however, the homomorphism equation (30) has no solutions for an Ansatz of the form (25)–(26). Thus, the best we can achieve is a representation of the algebra (1) on the quotient $\Psi\mathfrak{gl}_1/\mathcal{I}^{-3}$, where $\mathcal{I}^{-3}$ is the ideal of PSDOs of order ≤ -3.

On the other hand, it is easy to see from the definition (23) that $\tilde{c}_{\mathrm{R}}$ vanishes if one of its arguments lies in $\mathcal{I}^{-3}$. Since, furthermore, a nontrivial calculation shows that

$$\rho^* \tilde{c}_{\mathrm{R}}(X, Y) \equiv \tilde{c}_{\mathrm{R}}(\rho(X), \rho(Y)) = \omega_3(X, Y) \quad (32)$$

when ρ is given by (32), ρ generates the correct central term. Hence, we have obtained a homomorphism $\rho : \hat{\mathcal{L}}_3 \to \widetilde{\Psi\mathfrak{gl}_1}/\mathcal{I}^{-3}$, where the central extension $\widetilde{\Psi\mathfrak{gl}_1}$ of $\Psi\mathfrak{gl}_1$ is defined by the twisted Radul cocycle $\tilde{c}_{\mathrm{R}}$.

We now wish to second-quantize this representation. To this end, first note that the Fock space operators

$$\widehat{\rho(X)} = \sum_{m,n} \rho(X)_{mn} : a_m^* a_n : \quad (33)$$

obey the commutation relations of $\widehat{\mathfrak{gl}_1(\mathcal{H})}$ with the ill-defined Schwinger term $c_1(\rho(X), \rho(Y))$. Instead we define

$$\widehat{\widehat{\rho(X)}} = \widehat{\rho(X)} + \lambda(\rho(X)), \quad (34)$$

where λ is the one-cochain entering in eq. (24) [1]. From the fact that $\delta\lambda(S, S') = \lambda([S, S'])$ it follows that the operators (34) satisfy

$$[\widehat{\widehat{\rho(X)}}, \widehat{\widehat{\rho(Y)}}] = [\widehat{\rho(X)}, \widehat{\rho(Y)}] + \tilde{c}_{\mathrm{R}}(\rho(X), \rho(Y)), \quad (35)$$

where the Schwinger term is now well defined.

However, since

$$\widehat{\widehat{\rho(H)}}|0\rangle = \lambda(\rho(H))|0\rangle, \quad (36)$$

the original Fock vacuum is not a HW state for a representation of the algebra (35) on $\mathcal{F}(\mathcal{H})$. Unfortunately, we have not been able to find such a vacuum state. In addition, we need a way to characterize Fock space states that are annihilated by the ideal $\mathcal{I}^{-3}$ in order to be able to divide out this ideal from the representation.

6. Conclusions and comments

Although we have obtained a potentially interesting realization of a higher-dimensional loop algebra as a current algebra on a fermionic Fock space, it appears difficult to overcome the remaining obstacles to the construction of a unitary HW representation. On the other hand, it is fair to say that had our construction worked completely, it would have been a truly remarkable result.

In any case, the search for this representation has lead to some nice mathematical results [25], namely the proof of the cohomological equivalence on the algebra of second-quantizable PSDOs between the Lundberg (a.k.a. Kac–Peterson) cocycle defining the universal central extension of $\mathfrak{gl}_1(\mathcal{H})$ and a twisted version of the Radul cocycle. We also showed in ref. [25] that, in arbitrary number of dimensions d, the Radul cocycle may for two arbitrary PSDOs be written as the phase space integral of the order-$(-d)$ part of the star commutator of their symbols.

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